On Natural Deduction

W. V. Quine

Journal of Symbolic Logic, Volume 15, Issue 2 (Jun., 1950), 93-102.

THE JOURNAL OF SYMBOLIC LOGIC
Volume 15, Number 2, June 1950

ON NATURAL DEDUCTION

W. V. QUINE

1. General nature of natural deduction. For Gentzen's *natural deduction*, a formalized method of deduction in quantification theory dating from 1934,[1] these important advantages may be claimed: it corresponds more closely than other methods of formalized quantification theory to habitual unformalized modes of reasoning, and it consequently tends to minimize the false moves involved in seeking to construct proofs. The object of this paper is to present and justify a simplification of Gentzen's method, to the end of enhancing the advantages just claimed. No acquaintance with Gentzen's work will be presupposed.

A further advantage of Gentzen's method, also somewhat enhanced in my revision of the method, is relative brevity of proofs. In the more usual systematizations of quantification theory, theorems are derived from axiom schemata by proofs which, if rendered in full, would quickly run to unwieldy lengths. Consequently an abbreviative expedient is usually adopted which consists in preserving and numbering theorems for reference in proofs of subsequent theorems. Further brevity is commonly gained by establishing metatheorems, or derived rules, for reference in proving subsequent theorems. In natural deduction, on the other hand, proofs tend to be so short that the abbreviative expedients just now mentioned may conveniently be dispensed with—at least until theorems of extraordinary complexity are embarked upon. In natural deduction accordingly it is customary to start each argument from scratch, without benefit of accumulated theorems or derived rules.

It will be seen, when we proceed to details, that the rules of natural deduction lack certain traits of elegance which grace the more usual systematizations of quantification theory. Any reader who for this reason inclines to a systematization of the more usual kind may still look upon the rules of natural deduction (hereafter to be set forth) as a set of derived rules, or metatheorems, of his own system. Their derivability, granted the completeness of his own system, is assured by the ensuing proof of soundness (§6). The rules remain notable as metatheorems, in view of the virtues previously urged; and, as we shall see (§5), they are adequate to quantification theory in its entirety.

The *schemata* of quantification theory are built up of *atomic schemata* 'p', 'q', 'r', 'Fx', 'Fy', 'Gx', 'Hyz', etc., by means of truth-functional connectives '$\sim$', '$.$', '$\supset$', etc. and quantifiers '(x)', '(y)', '$(\exists x)$', etc. in the familiar ways. A schema is called *valid* if it comes out true no matter what truth values be assigned to 'p', 'q', etc., and no matter what non-empty universe be adopted as range for the variables 'x', 'y', etc., and no matter what classes and relations of objects in that universe be adopted as interpretations of 'F', 'G', 'H', etc., and no matter what objects in that universe be assigned to the free variables. One

Received April 18, 1949.

[1] Gerhard Gentzen, *Untersuchungen über das logische Schliessen*, ***Mathematische Zeitschrift***, vol. 39 (1934), pp. 176–210, 405–431.

schema is said to *imply* another if the conditional ('$\cdots \supset \cdots$') having the one schema as antecedent and the other as consequent is valid.

The system of natural deduction to be presented here is, like Gentzen's, primarily a method of establishing implication. Secondarily it is a method of establishing validity generally. A sample deduction (unintelligible, of course, pending ensuing explanations) is as follows:

* { (1)	$(\exists x)(y)(Fy \supset Gxy)$		
* { (2)	$(x)(y)(Gxy \supset Hyy)$		
* (3)	$(y)(Fy \supset Gwy)$	(1)	w
* (4)	$Fz \supset Gwz$	(3)	
* (5)	$(y)(Gwy \supset Hyy)$	(2)	
* (6)	$Gwz \supset Hzz$	(5)	
**(7)	Fz		
**(8)	Hzz	(4)(6)(7)	
**(9)	$(\exists y)Hzy$	(8)	
*(10)	$Fz \supset (\exists y)Hzy$	(9)	
*(11)	$(x)(Fx \supset (\exists y)Hxy)$.	(10)	z

A column of stars is a reminder that whatever is proved alongside it is dependent on the premiss, or the conjunction of the premisses, at which that column of stars begins. Thus the above deduction is a deduction of (11) from (1) and (2), and is supposed to establish that the conjunction of (1) and (2) implies (11). But we have yet to note the rules according to which the successive lines of such a deduction are set down.

2. Preliminary terminology. For stating the rules, this special terminology will be useful: *Inferring* a line (n) from a line (m) consists in writing (n) as a line later than the line (m) and alongside all the columns of stars (at least) which pass alongside the line (m), and appending to the line (n) a numeral referring back to (m). This definition of "inference" is a broad one, covering both good and bad inference; it is the business of the ensuing rules to say what is good. Without regard to the rules, however, we can look back to the illustrative deduction recorded above and note five cases which, for better or worse, are covered by our definition: (3) is inferred from (1), (4) from (3), (5) from (2), (6) from (5), (9) from (8), and (11) from (10).

The definition of inference needs, for some purposes, to be pluralized thus: *Inferring* a line (n) from *several* lines (i), (j), $\cdots$, (m) consists in writing (n) as a line later than all of (i), (j), $\cdots$, (m) and alongside all the columns of stars (at least) which pass alongside any of the lines (i), (j), $\cdots$, (m), and appending numerals to line (n) referring back to all of (i), (j), $\cdots$, (m). Thus, in the illustrative deduction set forth above, (8) may be said to be inferred from (4), (6), and (7).

Consider now any portion of a deduction, consisting say of consecutive lines (*h*) to (*k*), where (*h*) initiates a column of stars and (*k*) starts immediately alongside that column of stars (thus bearing no more stars than (*h*)). This whole body of lines from (*h*) to (*k*), including the stars in the column initiated by (*h*) and any stars to the right of them, but excluding any stars to the left, will be called a *subdeduction* of the given deduction. E.g., the portions:

*(7)	Fz	*(7)	Fz
*(8)	Hzz	*(8)	Hzz
*(9)	$(\exists y)Hzy$		

of the above example are subdeductions. *Inferring* a line (*n*) from a subdeduction consists in writing (*n*) as a line later than that subdeduction and alongside all the columns of stars (at least) which pass alongside that subdeduction (not counting the stars which are *in* the subdeduction), and appending a numeral to the line (*n*) referring back to the last line of that subdeduction. E.g., line (10) of our example is inferred from the subdeduction *(7)–*(9) singled out above.

If a line in a deduction stands alongside one or more columns of stars, all the lines at which any of those columns of stars begin will be called *premisses of* the line in question. Thus, in the foregoing example, (1) and (2) are premisses of every line; in addition (7) is a premiss of each of the lines (7)–(9).

Where α and β are any variables ('x', 'y', etc.), and ϕ is any schema in which α is free, the schema which is like ϕ except for containing free occurrences of β in place of all free occurrences of α will be called $\phi\frac{\beta}{\alpha}$.[2] In particular β may be free also in ϕ; and in particular β may be α, in which case $\phi\frac{\beta}{\alpha}$ is ϕ.

I shall use quasi-quotation marks as in previous writings. Thus, where ϕ and ψ are any unspecified schemata and α any unspecified variable, $\ulcorner\phi \supset \psi\urcorner$ is the conditional whose antecedent is ϕ and whose consequent is ψ; $\ulcorner(\exists\alpha)\phi\urcorner$ is the existential quantification of ϕ with respect to α; and so on.

3. The rules of the system may now be formulated as follows.

Rule of assumption (A): We may set down any schema as a line anywhere in the course of a deduction, provided that we initiate a new innermost column of stars at that point. This rule accounts for lines (1)–(2) and (7) of the illustrative deduction in §1. But lines (1)–(2) must be thought of, from the point of view of this rule, as written in conjunction to form a single line. Their separation into two lines, as in the above deduction, is an unofficial notational departure which involves tacit use of the following rule TF.

Rule of truth-functional inference (TF): From any line or lines we may infer any line which can be shown purely by truth-table considerations to be implied

[2] This notation, due to Whitehead and Russell, rests on a neat arithmetical analogy: $(mk)(n/m) = nk$. Confusingly, '$\phi(\alpha/\beta)$' is often written nowadays instead of '$\phi(\beta/\alpha)$.' This inversion probably rests on a pun between 'durch' in the fractional sense and 'α durch β ersetzen.'

by the given line or by the conjunction of the given lines. This rule accounts for (8) in the above example.

Rule of conditionalization (Cd): From any subdeduction whose initial assumption is ϕ and whose last line is ψ we may infer $\ulcorner \phi \supset \psi \urcorner$. This rule accounts for (10).

Rule of universal instantiation (UI): From $\ulcorner(\alpha)\phi\urcorner$ we may infer $\phi \frac{\beta}{\alpha}$. This rule accounts for lines (4)–(6).

Rule of existential generalization (EG): From $\phi \frac{\beta}{\alpha}$ we may infer $\ulcorner(\exists\alpha)\phi\urcorner$. This rule accounts for line (9).

Rule of universal generalization (UG): From $\phi \frac{\beta}{\alpha}$ we may infer $\ulcorner(\alpha)\phi\urcorner$ if β is alphabetically later than all free variables of $\ulcorner(\alpha)\phi\urcorner$. (Alphabetical order: w, x, y, z, w', x', y', z', w'', etc.) This rule accounts for line (11).

Rule of existential instantiation (EI): From $\ulcorner(\exists\alpha)\phi\urcorner$ we may infer $\phi \frac{\beta}{\alpha}$ if β is alphabetically later than all free variables of $\ulcorner(\exists\alpha)\phi\urcorner$. This rule accounts for line (3).

Flagging: Off to the right of any line inferred by UG or EI, the variable β must be written. Cf. lines (3) and (11).

Restriction: Neither UG nor EI is permissible if β has previously been flagged.

Also we must impose this restriction upon what are to be regarded as *finished deductions:* No flagged variable is to be free in the last line of a finished deduction, nor in any premiss of the last line.

Apart from notational details, four of the above rules are the same as in Gentzen's system: A, Cd, UI, and EG. TF takes the place of several rules of more elementary character in Gentzen's system. The various restrictions in connection with UI and EG are all new; these include the flagging notation, the restriction against flagging the same variable twice, the restriction on finished deductions, and the alphabetical stipulations in UG and EI. Gentzen has a rule which is like UG except that the restrictions governing it differ from mine. In place of EI, on the other hand, he has a very different rule, more complicated to state and less convenient to use.[3]

[3] Rule EI, my most conspicuous departure from Gentzen, is not altogether new. Several systems of natural deduction have been presented which differ from the present system and from one another in no essential way except in the restrictions adopted in connection with UG and EI. (They differ also in that in some of them TF is resolved into a bundle of more elementary rules; but this is a trivial matter.) Such a system was set forth by John C. Cooley, ***A primer of formal logic*** (Macmillan, 1942), pp. 126–140, but without exact formulation of restrictions. Another such system, exactly formulated, appeared in my mimeographed ***Short course in logic*** (Cambridge, Mass., 1946), but the restrictions here were insufficient. (For a fallacious result which can be obtained in that system, see the ten-line example in the next section.) A revised version appeared in my mimeographed ***Theory of deduction*** (Cambridge, 1948), but here, as J. W. Oliver has pointed out, UG was restricted beyond necessity and convenience. The present system is superior both practically and aesthetically to that of ***Theory of deduction.*** I have lately learned that Barkley Rosser has had, since 1940, an exactly formulated system of natural deduction which perhaps resembles the present system more closely than any of the others cited above. He set it forth in some mimeo-

If we construe '($\exists x$)' as an abbreviation of '$\sim(x)\sim$', we can dispense with EG and EI. A system which does just this was indeed published by Jaśkowski[4] simultaneously with Gentzen's publication. However, ease and naturalness of deduction are largely sacrificed when Jaśkowski's economy is imposed; so it is preferable to preserve the autonomy of existential quantification and keep all seven rules. The use of columns of stars is more reminiscent of Jaśkowski's notation than Gentzen's. Its specific form is due to a suggestion by Dr. Hao Wang.

4. Indispensability of the restrictions. Since alphabetical order seems irrelevant to logic, the reader may wonder at the alphabetical stipulations in UG and EI. Actually these stipulations are an artificial device for ruling out certain sequences of steps which, as the following two examples show, could lead to wrong results.

*(1)	$(x)(\exists y)Fxy$		(A)
*(2)	$(\exists y)Fwy$	(1)	(UI)
*(3)	Fwz	(2) z	(EI)
*(4)	$(x)Fxz$	(3) w	(UG fallaciously)
*(5)	$(\exists y)(x)Fxy$	(4)	(EG)

*(1)	$(x)(\exists y)Fxy$		(A)
*(2)	$(\exists y)Fzy$	(1)	(UI)
*(3)	Fzw	(2) w	(EI fallaciously)
*(4)	$(x)Fxw$	(3) z	(UG)
*(5)	$(\exists y)(x)Fxy.$	(4)	(EG)

It is well known that (1) does not in fact imply (5); e.g., take 'F' as identity.

An incidental effect of the alphabetical stipulation in UG and EI is that it forbids free occurrences of β in $\ulcorner(\alpha)\phi\urcorner$ and $\ulcorner(\exists\alpha)\phi\urcorner$. Wrong results of the following two kinds are thereby obviated.

*(1)	$(x)Fxx$		(A)
*(2)	Fyy	(1)	(UI)
*(3)	$(x)Fxy$	(2) y	(UG fallaciously)
*(4)	$(\exists y)(x)Fxy$	(3)	(EG)

*(1)	$(x)(\exists y)Fxy$		(A)
*(2)	$(\exists y)Fxy$	(1)	(UI)
*(3)	Fxx	(2) x	(EI fallaciously)
*(4)	$(\exists x)Fxx.$	(3)	(EG)

Actually '$(x)Fxx$' does not imply '$(\exists y)(x)Fxy$', as may be seen by construing 'F' as identity; nor does '$(x)(\exists y)Fxy$' imply '$(\exists x)Fxx$', as may be seen by construing 'F' as diversity.

---

graphed lecture notes in 1946–47. Having learned of this during travels in which I am still engaged, I do not yet know the details of his system. I was influenced in latter-day revisions of my present system, however, by information that Rosser's UG and EI were symmetrical to each other.

[4] Stanisław Jaśkowski, *On the rules of suppositions in formal logic*, **Studia logica**, no. 1 (Warsaw, 1934; 32 pp.).

The restriction against flagging the same variable twice is directed against fallacies of the following sorts:

*(1)	$(\exists x)Fx$		(A)
*(2)	Fy	(1) y	(EI)
*(3)	$(x)Fx$	(2) y	(UG fallaciously)

*{(1)	$(\exists x)Fx$		(A)
*{(2)	$(\exists x)Gx$		
*(3)	Fy	(1) y	(EI)
*(4)	Gy	(2) y	(EI fallaciously)
*(5)	$Fy \,.\, Gy$	(3)(4)	(TF)
*(6)	$(\exists x)(Fx \,.\, Gx)$	(5)	(EG)

*(1)	Fz		(A)
*(2)	$(y)Fy$	(1) z	(UG)
(3)	$Fz \supset (y)Fy$	(2)	(Cd)
(4)	$(x)(Fx \supset (y)Fy)$.	(3) z	(UG fallaciously)

Line (4) here, being unstarred, pretends to validity; but it is not valid, being in fact equivalent to '$(\exists x)Fx \supset (y)Fy$'.

*(1)	$(x)(\exists z)(Fxz \,.\, Gz \supset Gx)$		(A)
*(2)	$(\exists z)(Fxz \,.\, Gz \supset Gx)$	(1)	(UI)
*(3)	$Fxz \,.\, Gz \supset Gx$[5]	(2) z	(EI)
**(4)	Gz		(A)
**(5)	Gx	(3)(4)	(TF)
**(6)	$(y)Gy$	(5) x	(UG)
*(7)	$Gz \supset (y)Gy$	(6)	(Cd)
*(8)	$Fxz \,.\, Gz \supset (y)Gy$	(3)(7)	(TF)
*(9)	$(\exists z)(Fxz \,.\, Gz \supset (y)Gy)$	(8)	(EG)
*(10)	$(x)(\exists z)(Fxz \,.\, Gz \supset (y)Gy)$.	(9) x	(UG fallaciously)

That (1) does not really imply (10) can be seen by construing 'F' as identity and 'G' as redness. Thereupon (1) becomes true and (10) false, for any universe containing some red and some non-red things.

The requirement that no flagged variable be free in the last line restrains us from viewing either of these as a finished deduction:

*(1)	$(\exists x)Fx$	
*(2)	Fy,	(1) y

*(1)	Fy	
*(2)	$(x)Fx$	(1) y
(3)	$Fy \supset (x)Fx$.	(2)

Actually of course (1) implies (2) in neither example, nor is (3) valid.[6] Similarly

---

[5] Readers accustomed to dot conventions at variance with Whitehead and Russell's should be warned that this means '$Fxz \,.\, (Gz \supset Gx)$.'

[6] The definition of implication as validity of the conditional must be borne in mind. To deny that 'Fy' implies '$(x)Fx$' is not to deny that '$(y)Fy$' implies '$(x)Fx$,' but it is to deny rather that '$Fy \supset (x)Fx$' is valid, or, in other words, that '$(y)(Fy \supset (x)Fx)$' is valid.

the requirement that no flagged variable be free in a premiss of the last line restrains us from viewing either of these as a finished deduction:

*(1)	Fy	
*(2)	$(x)Fx$,	(1) y

*{(1)	$(\exists x)Fx$	
*{(2)	Gy	
*(3)	Fy	(1) y
*(4)	$Fy \,.\, Gy$	(2)(3)
*(5)	$(\exists x)(Fx \,.\, Gx)$.	(4)

5. Completeness of the system will now be established by showing that everything demonstrable in the classical version of quantification theory can be reached as an unstarred last line of a finished deduction in our system; for, it is known that every valid schema is demonstrable in the classical version of quantification theory.[7] In what I am calling the classical version, theorem schemata are derived from the axiom schemata

(a) $(x)Fx \supset Fy$, (b) $Fy \supset (\exists x)Fx$,

together with various truth-functionally valid schemata, by these rules:

(I) From $\ulcorner \phi \supset \psi \urcorner$ draw $\ulcorner \phi \supset (\alpha)\psi \urcorner$, if α is not free in ϕ;

(II) From $\ulcorner \phi \supset \psi \urcorner$ draw $\ulcorner (\exists\alpha)\phi \supset \psi \urcorner$, if α is not free in ψ,

together with familiar rules of substitution, modus ponens, and relettering of bound variables.

Now the axiom schema (a) can be proved by means of our rules thus:

*(1)	$(x)Fx$		(A)
*(2)	Fy	(1)	(UI)
(3)	$(x)Fx \supset Fy$.	(2)	(Cd)

Similarly for (b), using EG instead of UI. Any truth-functionally valid schema can be proved by inferring it arbitrarily from any prior unstarred line, say (3) above, by TF. Moreover, whatever can be got from proved schemata by means of modus ponens can be got from them in our system by TF. Again, if a schema ψ can be got from a proved schema ϕ by substitution, or by relettering bound variables, then ψ can be proved in our system simply by repeating the proof of ϕ itself with the appropriate substitutions or changes of letters introduced throughout. In thus relettering the proof of ϕ we may have to advance all flagged variables alphabetically, in order to preserve conformity with the alphabetical stipulations in UG and EI; but this presents no difficulty, the alphabet being endless. Such adjustment of flagged variables will not alter the choice of variables in ψ itself, since flagged variables never have free occurrences in last lines of finished deductions.

So now only (I) and (II) remain to be reckoned with. What we want to show, so far as (I) is concerned, is that if α is free in ψ and not in ϕ, and $\ulcorner \phi \supset \psi \urcorner$ can be proved (i.e., can be reached as unstarred last line of a finished deduction),

[7] Kurt Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, ***Monatshefte für Mathematik und Physik***, vol. 37 (1930), pp. 349–360.

then so can $\ulcorner\phi \supset (\alpha)\psi\urcorner$. Let β be any variable alphabetically later than those of ϕ and ψ. Now let us reletter the proof of $\ulcorner\phi \supset \psi\urcorner$, changing each free α to β and then uniformly advancing all flagged variables alphabetically as far as necessary in order to preserve conformity with the alphabetical stipulation in UG and EI. Since the proof of $\ulcorner\phi \supset \psi\urcorner$ was a finished deduction, no flagged variable was free in $\ulcorner\phi \supset \psi\urcorner$; in the relettered proof, therefore, the only change of variable in the last line is the change of α to β. We thus have a proof of $\ulcorner\phi \supset \psi \frac{\beta}{\alpha}\urcorner$. But we can extend it to a proof of $\ulcorner\phi \supset (\alpha)\psi\urcorner$ by adding four lines as follows:

$\ulcorner(k)$	$\phi \supset \psi \frac{\beta}{\alpha}$		
$*(k + 1)$	ϕ		
$*(k + 2)$	$\psi \frac{\beta}{\alpha}$	$(k)(k + 1)$	
$*(k + 3)$	$(\alpha)\psi$	$(k + 2)$	β
$(k + 4)$	$\phi \supset (\alpha)\psi$	$(k + 3)\urcorner$.	

As for (II), what we want to show is that if α is free in ϕ and not in ψ, and $\ulcorner\phi \supset \psi\urcorner$ can be proved, then so can $\ulcorner(\exists\alpha)\phi \supset \psi\urcorner$. Similarly to what was seen in the preceding paragraph, we can get a proof of $\ulcorner\phi \frac{\beta}{\alpha} \supset \psi\urcorner$; and then we can extend it to a proof of $\ulcorner(\exists\alpha)\phi \supset \psi\urcorner$ by adding four lines as follows:

$\ulcorner(k)$	$\phi \frac{\beta}{\alpha} \supset \psi$		
$*(k + 1)$	$(\exists\alpha)\phi$		
$*(k + 2)$	$\phi \frac{\beta}{\alpha}$	$(k + 1)$	β
$*(k + 3)$	ψ	$(k)(k + 2)$	
$(k + 4)$	$(\exists\alpha)\phi \supset \psi$.	$(k + 3)\urcorner$	

6. Soundness of the system remains to be established. I.e., what is to be shown is that *the last line of any finished deduction is implied by its premisses.* 'Implied by its premisses' is understood as meaning 'implied by the conjunction of its premisses' if there are many, 'implied by its premiss' if there is but one, and 'valid' if there are none.

LEMMA 1: If in a deduction devoid of flagged variables each line prior to a line (n) is implied by its own premisses, then so is (n).

Proof. (n) is got by A, TF, UI, EG, or Cd.

Case A: Trivial; (n) is one of its own premisses.

Case TF: Here (n) is inferred from a prior line or lines (m_1), (m_2), $\cdots$, (m_k) which imply (n). All premisses of (m_1), $\cdots$, (m_k) are premisses of (n), by the definition of inference. But, by hypothesis, each of (m_1), $\cdots$, (m_k) is implied by its own premisses. Therefore the premisses of (n) imply (m_1), $\cdots$, (m_k), which in turn imply (n).

Cases UI and EG: Here, since $\ulcorner(\alpha)\phi\urcorner$ implies $\phi\frac{\beta}{\alpha}$ and $\phi\frac{\beta}{\alpha}$ implies $\ulcorner(\exists\alpha)\phi\urcorner$, the argument proceeds as in Case TF with $k = 1$.

Case Cd: (n) is $\ulcorner\phi \supset \psi\urcorner$, and is inferred from a subdeduction which starts with ϕ and ends with ψ. By hypothesis, the premisses of ψ imply ψ. Hence the premisses of ψ other than ϕ imply $\ulcorner\phi \supset \psi\urcorner$, or (n). But, by the definition of inference from a subdeduction, those premisses are premisses of (n).

LEMMA 2: In deductions without flagged variables each line is implied by its premisses.

Proof. The first line is implied by its premisses, being itself a premiss. By Lemma 1, then, it follows inductively that each line is implied by its premisses.

LEMMA 3: If the last line of every finished deduction with just n flagged variables is implied by its premisses, then the last line of every finished deduction with just $n + 1$ flagged variables is implied by its premisses.

Proof. Let D be a finished deduction with $n + 1$ flagged variables, the alphabetically first of which is β, flagged say in line (k). This line and some earlier one, say (h), are then related in one or the other of these ways:

$$\begin{array}{llll} \ulcorner\ldots{**}(h) & \phi\frac{\beta}{\alpha} & & \\ \ldots{**}(k) & (\alpha)\phi & (h) & \beta\urcorner \end{array} \quad \text{or} \quad \begin{array}{llll} \ulcorner\ldots{**}(h) & (\exists\alpha)\phi & & \\ \ldots{**}(k) & \phi\frac{\beta}{\alpha} & (h) & \beta\urcorner. \end{array}$$

Form now a revised deduction D' by prefixing to D a premiss (0), of the form of a conditional with (h) as antecedent and (k) as consequent:

$$\ulcorner{*}(0) \qquad \phi\frac{\beta}{\alpha} \supset (\alpha)\phi\urcorner \qquad \text{or} \qquad \ulcorner{*}(0) \qquad (\exists\alpha)\phi \supset \phi\frac{\beta}{\alpha}\urcorner.$$

In D' we can unflag β, since (k) is now justifiable by TF on the basis of (h) and (0). All other lines of D' remain as in D, except that a new column of stars, initiated by (0), runs through to the end of the deduction. Now, supposing (m) to be the last line of D and of D', let us record sundry observations:

(i) β is not free in (m). For D was a finished deduction.

(ii) β is free in no premiss of (m) in D. Same reason.

(iii) No other flagged variable of D is free in (m) nor in any premiss of (m) in D. Same reason.

(iv) Of the flagged variables of D, none but β is free in $\phi\frac{\beta}{\alpha}$. For, by UG and EI, β is alphabetically the last free variable of $\phi\frac{\beta}{\alpha}$; yet β is alphabetically the first flagged variable of D.

(v) No flagged variable of D' is free in (0). By (iv), since β is unflagged in D'.

(vi) D' is a finished deduction. By (iii) and (v).

(vii) (m) is implied by its premisses in D'. By (vi) and the hypothesis of Lemma 3.

(viii) (m) is implied by (0) and the premisses of (m) in D. By (vii).

(ix) (m) is implied by its premisses in D together with $\ulcorner(\exists\beta)\left(\phi\frac{\beta}{\alpha}\supset(\alpha)\phi\right)\urcorner$ or $\ulcorner(\exists\beta)\left((\exists\alpha)\phi\supset\phi\frac{\beta}{\alpha}\right)\urcorner$. By (i), (ii), and (viii).

(x) These latter two quantifications are valid. For, they are equivalent by classical quantification theory to $\ulcorner(\beta)\phi\frac{\beta}{\alpha}\supset(\alpha)\phi\urcorner$ and $\ulcorner(\exists\alpha)\phi\supset(\exists\beta)\phi\frac{\beta}{\alpha}\urcorner$. (It is essential here that β not be free in $\ulcorner(\alpha)\phi\urcorner$ or $\ulcorner(\exists\alpha)\phi\urcorner$, but this is assured by the statement of UG and EI.)

From (ix) and (x) it follows that (m) is implied by its premisses in D; and this completes the proof of Lemma 3.

THEOREM: The last line of every finished deduction is implied by its premisses. *Proof* by induction from Lemmas 2 and 3.

HARVARD UNIVERSITY